On the coupling of Spectral Element Method with Discontinuous Galerkin approximation for elasto-acoustic problems.

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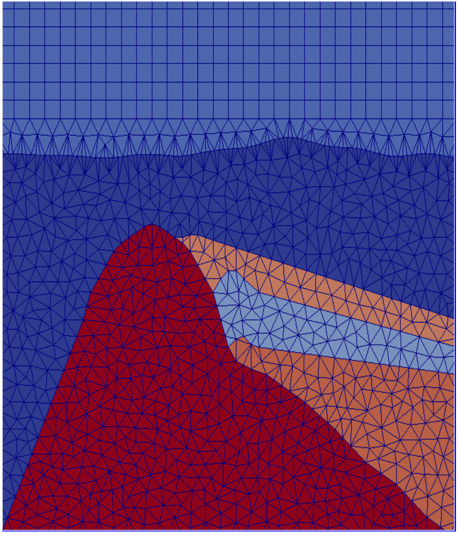
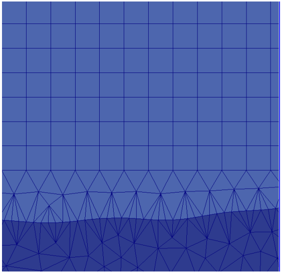
Why using hybrid meshes?





- Useful when the use of unstructured grid is non-sense (e.g. medium with a layer of water)
- Allows the coupling of numerical methods in order to reduce the computational cost

Elastodynamic system

$x \in \Omega \subset \mathbb{R}^d$, $t \in [0, T]$, $T > 0$:

$$\begin{cases} \rho(x)\dfrac{\partial v}{\partial t}(x,t) & = \nabla \cdot \underline{\sigma}(x,t) \\ \dfrac{\partial \underline{\underline{\sigma}}}{\partial t}(x,t) & = \underline{\underline{C}}(x)\underline{\underline{\epsilon}}(v(x,t)) \end{cases}$$

With :

- $\rho(x)$ the density
- $C(x)$ the elasticity tensor
- $\epsilon(x,t)$ the deformation tensor
- $v(x,t)$, the wavespeed
- $\underline{\sigma}(x,t)$ the strain tensor

Elasticus software

Software written in **Fortran 90** for wave propagation simulation in the **time domain**

Features

Simulation:

- on various types of meshes (**unstructured triangle, structured quadrangle, hybrid**)
- on **heterogeneous** media (**acoustic, elastic and elasto-acoustic**)

- **Discontinuous Galerkin** (DG) on **quadrangle, triangle and hybrid mesh**
- **Spectral Element Method** (SEM) only on **quadrangle mesh**
- with various time-schemes : **Runge-Kutta (2 or 4), Leap-Frog**
- with **p-adaptivity**, **multi-order** computation...

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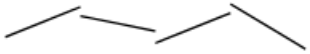
1. Numerical Methods
    - Discontinous Galerkin Method (DG)
    - Spectral Element Method (SEM)
    - Advantages of each method

Use discontinuous functions :

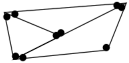
mesh

continuous

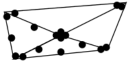


discontinuous

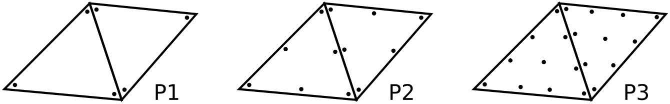
h adaptivity :



p adaptivity :

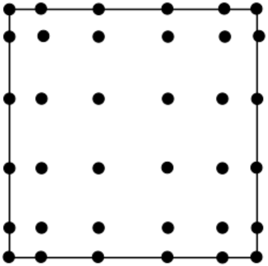


Degrees of freedom necessary on each cell :



General principle

- Finite Element Method (FEM) discretization + Gauss-Lobatto quadrature
- Gauss-Lobatto points as degrees of freedom (gives us exponential convergence on L^2-norm)



- $\int f(x)dx \approx \sum_{j=1}^{N+1} \omega_j f(\xi_j)$
- $\varphi_i(\xi_j) = \delta_{ij}$

Spectral Element Method

Main change with DG

- DG discontinuous, SEM continuous
- Need to define local to global numbering
- Global matrices needed for SEM
- Basis functions computed differently

Advantages of each method

DG

- Element per element computation (*hp*-adaptivity)
- Time discretization quasi explicit (block diagonal mass matrix)
- Simple to parallelize

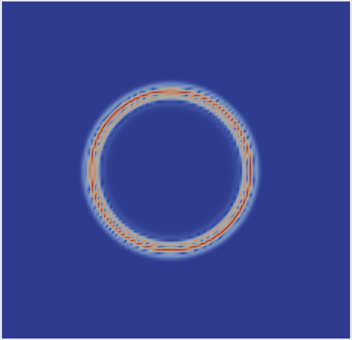
SEM

- Couples the flexibility of FEM with the accuracy of the pseudo-spectral method
- Reduces the computational cost when you use structured meshes in comparison with DG
- Simplifies the mass and stiff matrices (mass matrix diagonal)

2. Comparison DG/SEM on structured quadrangle mesh
   - Description of the test cases
   - Comparative tables

Description of the test cases

Physical parameters



P wavespeed	$1000\ m.s^{-1}$
Density	$1\ kg.m^{-3}$

Second order **Ricker Source** in *Pwave* ($f_{peak} = 10Hz$)

General context

- **Acoustic homogeneous** medium
- Four different meshes : **10000 cells, 22500 cells**, 90000 cells, 250000 cells
- CFL computed using **power iteration** method
- **Leap-Frog** time scheme
- **Four threads** parallel execution with **OpenMP**

Comparative tables

- Comparison between numerical solution and analytical solution obtained using the software Gar6more

Quadrangle mesh 10000 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.99e-3	2.5e-2	61.96	500
SEM	4.9e-3	1.3e-1	0.73	204
SEM(DG CFL)	1.99e-3	4.8e-2	1.48	502

Quadrangle mesh 22500 elements:

	CFL	L2-error	CPU-time	Nb of time steps
DG	1.33e-3	1.8e-2	252.20	750
SEM	3.26e-3	7e-2	2.42	306
SEM(DG CFL)	1.33e-3	1.2e-2	4.70	751

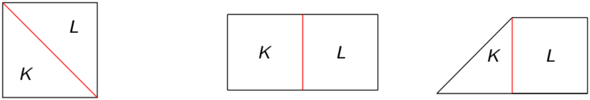
SEM fifty time much faster on a mesh with 22500 cells than DG

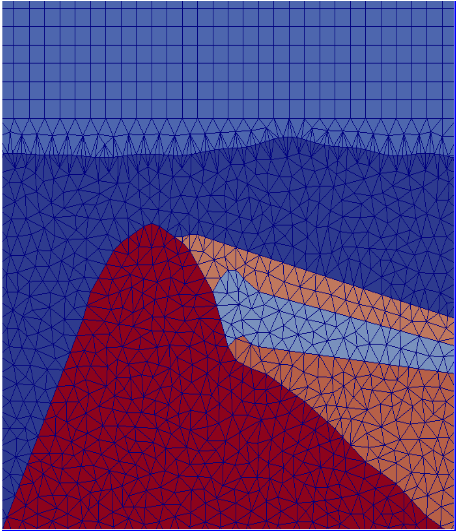
3. DG/SEM coupling
   - Hybrid meshes structures
   - Variationnal formulation
   - Space discretization

Hybrid meshes structures

- Need to couple P_k and Q_k structures.
- Need to extend or split some of the structures (e.g. neighbour indexes)
- Necessity to define new face matrices

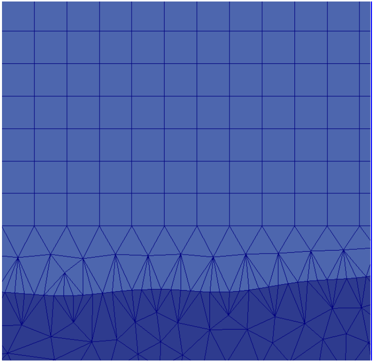
$$M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \phi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \psi_i^K \psi_j^L, \quad M_{ij}^{K,L} = \int_{K \cap L} \phi_i^K \psi_j^L$$





Global context

- Domain in two parts : $\Omega_{h,1}$ (**structured quadrangle + SEM**), $\Omega_{h,2}$ (**unstructured triangle + DG**)
- w_1,w_2 the tests-function and ξ_1, ξ_2 the tests-tensors



SEM variationnal formulation :

$$\begin{cases} \displaystyle\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out}} (\sigma_1 n_1) \cdot w_1 \\ \displaystyle\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = -\int_{\Omega_{h,1}} (\nabla (C\xi_1)) \cdot v_1 + \int_{\Gamma_{out}} (C\xi_1 n_1) \cdot v_1 \end{cases}$$

DG variationnal formulation :

$$\begin{cases} \displaystyle\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot n_2 \\ \displaystyle\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,2}} (\nabla (C\xi_2)) \cdot v_2 + \int_{\Gamma_{out}} (C\xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C\xi_2]] \cdot n_2 \end{cases}$$

SEM variationnal formulation :

$$\begin{cases} \displaystyle\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \int_{\Gamma_{out}} (\sigma_1 n_1) \cdot w_1 + \int_{\Gamma_{DG\,/\,SEM}} (\sigma_1 n_1) \cdot w_1 \\ \displaystyle\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = -\int_{\Omega_{h,1}} (\nabla(C\xi_1)) \cdot v_1 + \int_{\Gamma_{out}} (C\xi_1 n_1) \cdot v_1 + \int_{\Gamma_{DG\,/\,SEM}} (C\xi_1 n_1) \cdot v_1 \end{cases}$$

DG variationnal formulation :

$$\begin{cases} \displaystyle\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{out}} (\sigma_2 n_2) \cdot w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot n_2 + \int_{\Gamma_{DG\,/\,SEM}} (\sigma_2 n_2) \cdot w_2 \\ \displaystyle\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,2}} (\nabla(C\xi_2)) \cdot v_2 + \int_{\Gamma_{out}} (C\xi_2 n_2) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C\xi_2]] \cdot n_2 + \int_{\Gamma_{DG\,/\,SEM}} (C\xi_2 n_2) \cdot v_2 \end{cases}$$

Computation steps

1. Simplify the coupling terms and separates the two parts + put $\sigma \cdot n = 0$

$$\begin{cases} \displaystyle\int_{\Omega_{h,1}} \rho \partial_t v_1 \cdot w_1 = -\int_{\Omega_{h,1}} \sigma_1 \cdot \nabla w_1 + \frac{1}{2}\int_{\Gamma_{DG/SEM}} (\sigma_1 + \sigma_2) n_1 \cdot w_1 \\ \displaystyle\int_{\Omega_{h,1}} \partial_t \sigma_1 : \xi_1 = -\int_{\Omega_{h,1}} (\nabla (C\xi_1)) \cdot v_1 + \frac{1}{2}\int_{\Gamma_{DG/SEM}} (C\xi_1 n_1) \cdot (v_1 + v_2) \end{cases}$$

$$\begin{cases} \displaystyle\int_{\Omega_{h,2}} \rho \partial_t v_2 \cdot w_2 = -\int_{\Omega_{h,2}} \sigma_2 \cdot \nabla w_2 + \int_{\Gamma_{int}} \{\{\sigma_2\}\}[[w_2]] \cdot n_2 - \frac{1}{2}\int_{\Gamma_{DG/SEM}} (\sigma_1 + \sigma_2) n_1 \cdot w_2 \\ \displaystyle\int_{\Omega_{h,2}} \partial_t \sigma_2 : \xi_2 = -\int_{\Omega_{h,2}} (\nabla (C\xi_2)) \cdot v_2 + \int_{\Gamma_{int}} \{\{v_2\}\}[[C\xi_2]] \cdot n_2 - \frac{1}{2}\int_{\Gamma_{DG/SEM}} (C\xi_2 n_1) \cdot (v_1 + v_2) \end{cases}$$

- φ_i : SEM basis functions
- ψ_i : DG basis functions

$$\begin{cases} M_{\mathbf{v}_1}\partial_t \mathbf{v}_{h,1} + R_{\boldsymbol{\sigma}_1}\boldsymbol{\sigma}_{h,1} + R^{2,1}_{\boldsymbol{\sigma}_2}\boldsymbol{\sigma}_{h,2} = 0 \\ M_{\boldsymbol{\sigma}_1}\partial_t \boldsymbol{\sigma}_{h,1} + R_{\mathbf{v}_1}\mathbf{v}_{h,1} + R^{2,1}_{\mathbf{v}_2}\mathbf{v}_{h,2} = 0 \end{cases}$$

- $M_{ij} = \int_\Omega \varphi_i\varphi_j \approx \sum_{e\in supp(\varphi_i)\cap supp(\varphi_j)} \sum_{k=1}^{(r+1)^d} \omega_k\varphi_i(\xi_k)\varphi_j(\xi_k) = \sum_{e\in supp(\varphi_i)\cap supp(\varphi_j)} \omega_i\delta_{i,j}$ the mass matrix
- $R_{p_{ij}} = \int_\Omega \varphi_i \frac{\partial\varphi_j}{\partial p}$ stiffness matrix

Matrix of DG/SEM coupling :

$$R^{2,1}_{\sigma_2,ij} = \int_{\partial\Omega_1\cap\partial\Omega_2} \psi_i\varphi_j$$

Space discretization : DG part

$$\begin{cases} \rho M_{\mathbf{v}_2} \partial_t \mathbf{v}_{h,2} + R_{\boldsymbol{\sigma}_2} \boldsymbol{\sigma}_{h,2} - R^{1,2}_{\boldsymbol{\sigma}_1} \boldsymbol{\sigma}_{h,1} = 0 \\ M_{\boldsymbol{\sigma}_2} \partial_t \boldsymbol{\sigma}_{h,2} + R_{\mathbf{v}_2} \mathbf{v}_{h,2} - R^{1,2}_{\mathbf{v}_1} \mathbf{v}_{h,1} = 0 \end{cases}$$

- $M_{ij}^K = \int_K \psi_i^K \psi_j^K$ mass matrix,
- $R_{p_{ij}}^K = \int_K \psi_i^K \dfrac{\partial \psi_j^K}{\partial p}$ stiffness matrix,
- $R_{p_{ij}}^{K,L} = \int_{\partial K \cap \partial L} \psi_i^K \psi_j^L n_K \cdot e_p$ the mass-face matrices

Two new matrices which come from the DG/SEM hybridation $R_\star^{1,2}$. Block composed :

$$R_{v_1}^{1,2} = R_{\sigma_1}^{1,2} = -\frac{1}{2} \int_{\partial \Omega_2 \cap \partial K_1} \psi_j^{K_2} \varphi_i \quad \forall i, j = 1..N_m \tag{1}$$

Conclusion and perspectives

Conclusion

1. As expected, SEM is more efficient on structured quadrangle mesh than DG
2. Show the utility on the use of hybrid meshes and method coupling (reduce computational cost,...)
3. Build a variationnal formulation for DG/SEM coupling and find a CFL condition that ensures stability

Perspectives

- Implement DG/SEM coupling on the code (2D)
- Develop h-adaptivity for the structured part
- Develop DG/SEM coupling in 3D

Thank you for your attention !

Questions?